Slides for the course

Statistics and econometrics

Appendix: some basic asymptotic results

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Outline

Convergence in probability,
Weak Law of Large Numbers
and Continous Mapping Theorem

Convergence in distribution,
Central Limit Theorem
and Slutzky Theorem

Delta method

A basic list of useful asymptotic results

We do not have time for a proper treatment of asymptotic theory in this short course.

Here we just report some basic results, used in the lecture notes, concerning:

- Concepts of convergence
- Laws of large numbers
- Limit theorems
- Other useful theorems for asymptotic calculus

For proofs, see Casella and Berger

A note on “how large” is “large”

We will never have a sample of size $n = \infty$. Are then asymptotic results useless?

No: the good properties of asymptotic results may be achieved in practice even in samples of finite size $n < \infty$.

This is the reason why asymptotic results are useful even if it is obvious that we will never have an infinite sample.

A finite sample may be sufficiently “large” for asymptotic results to hold with a very good approximation, even if its size is effectively not so large.

Section 1

Convergence in probability, Weak Law of Large Numbers and Continous Mapping Theorem

Definition of convergence in probability

A sequence of random variables X_n converges in probability to a random variable X if

$$\lim_{n \to +\infty} Pr(|X_n - X| > \epsilon) = 0 \qquad \forall \epsilon \tag{1}$$

Equivalent notations to denote convergence in probability are

$$X_n \xrightarrow{p} X \tag{2}$$

$$\underset{n \to +\infty}{\text{Plim}} X_n = X \tag{3}$$

Continuous mapping theorems for P-convergence

1. For any random variable X_n and continuous function $h(.)$:

$$\underset{n\to+\infty}{\text{Plim}} X_n = X \quad \Rightarrow \quad \underset{n\to+\infty}{\text{Plim}} h(X_n) = h(X) \tag{4}$$

2. Given two random variables such that

$$X_n \xrightarrow{p} X \quad \text{and} \quad Z_n \xrightarrow{p} Z \tag{5}$$

then

$$\underset{n\to+\infty}{\text{Plim}} (X_n + Z_n) = X + Z \tag{6}$$

$$\underset{n\to+\infty}{\text{Plim}} (X_n Z_n) = XZ \tag{7}$$

$$\underset{n\to+\infty}{\text{Plim}} \left(\frac{X_n}{Z_n}\right) = \frac{X}{Z} \tag{8}$$

Weak Law of Large Numbers

Consider a sample of iid random variables $\{X_1, ..., X_n\}$ with

$$E(X_i) = \mu \qquad \text{and} \qquad Var(X_i) = \sigma^2 \tag{9}$$

then

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \xrightarrow{p} \mu \tag{10}$$

The proof is a straightforward application of Chebychev inequality

The WLLN states that under fairly general conditions, sample moments converge in probability to population moments.

See Casella and Berger for the concept of *Almost sure* convergence (less relevant for econometrics) and the correspond Strong Law of Large Numbers

Section 2

Convergence in distribution, Central Limit Theorem and Slutzky Theorem

Definition of convergence in distribution

A sequence of random variables X_n converges in distribution to a random variable X if

$$\lim_{n \to +\infty} F_{X_n}(x) = F_X(x) \qquad \Longleftrightarrow \qquad X_n \xrightarrow{d} X \tag{11}$$

for all values x for which the Cumulative Distribution Function $F_X(x)$ is continuous.

It can be shown that Convergence in Probability implies Convergence in Distribution but the converse is not true

$$X_n \xrightarrow{p} X \qquad \Rightarrow \qquad X_n \xrightarrow{d} X \tag{12}$$

However, if $X_n \xrightarrow{d} C$ and C is a constant, then

$$X_n \xrightarrow{d} C \qquad \Rightarrow \qquad X_n \xrightarrow{p} C \tag{13}$$

Mapping theorems for D-convergence

1. Continuous Mapping: for any random variable X_n and continuous function $h(.)$:

$$X_n \xrightarrow{d} X \quad \Rightarrow \quad h(X_n) \xrightarrow{d} h(X) \tag{14}$$

2. Slutzky: Given two random variables such that

$$X_n \xrightarrow{d} X \quad \text{and} \quad Z_n \xrightarrow{p} C \tag{15}$$

where C is a constant, then

$$(X_n + Z_n) \xrightarrow{d} X + C \tag{16}$$

$$(X_n Z_n) \xrightarrow{d} XC \tag{17}$$

$$\left(\frac{X_n}{Z_n}\right) \xrightarrow{d} \frac{X}{C} \tag{18}$$

Central Limit Theorem

Consider a sample of iid random variables $\{X_1, ..., X_n\}$ with

$$E(X_i) = \mu \qquad \text{and} \qquad Var(X_i) = \sigma^2 < \infty. \tag{19}$$

Let

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n \qquad \text{and} \qquad \sqrt{n}(\bar{X}_n - \mu) \sim G_n(x) \tag{20}$$

Then

$$\lim_{n\to+\infty} G_n(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{21}$$

i.e. $\sqrt{n}(\bar{X}_n - \mu)$ converges to a normal distribution with zero mean and variance equal to σ^2.

$$\sqrt{n}(\bar{X}_n - \mu) \quad \xrightarrow{d} \quad N(0, \sigma^2) \tag{22}$$

Why is the CLT so crucially important for us?

Starting from an iid random sample,

- without making any distributional assumption,

the theorem states that moments of the sample are distributed according to a Standardised Normal, after

- subtracting the moment's mean,
- dividing for the moment's standard deviation
- and multiplying for the root square of the sample size

This result is extremely useful and powerful because it allows us

- to characterize the distribution of sample statistics (in particular large sample test statistics),
- even when we know nothing about the distribution of the random variables from which the sample has been drawn.

Section 3

Delta method

D-Convergence of random variable transformations

Consider a sequence of random variables X_n such that

$$\sqrt{n}(X_n - \theta) \xrightarrow{d} \text{Normal}(0, \sigma^2) \tag{23}$$

For any given function $g(.)$ and a specific value of θ, suppose that $g'(\theta)$ exists and is not 0. Then:

$$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{d} \text{Normal}(0, \sigma^2(g'(\theta))^2)) \tag{24}$$

This result is crucial to characterise the asymptotic distribution of a transformation of a test statistic.

Pay attention to the difference between the Delta Method result and the Continuous Mapping theorem for D-convergence, which says that

$$g\left(\sqrt{n}(X_n - \theta)\right) \xrightarrow{d} g\left(\text{Normal}(0, \sigma^2)\right) \tag{25}$$