

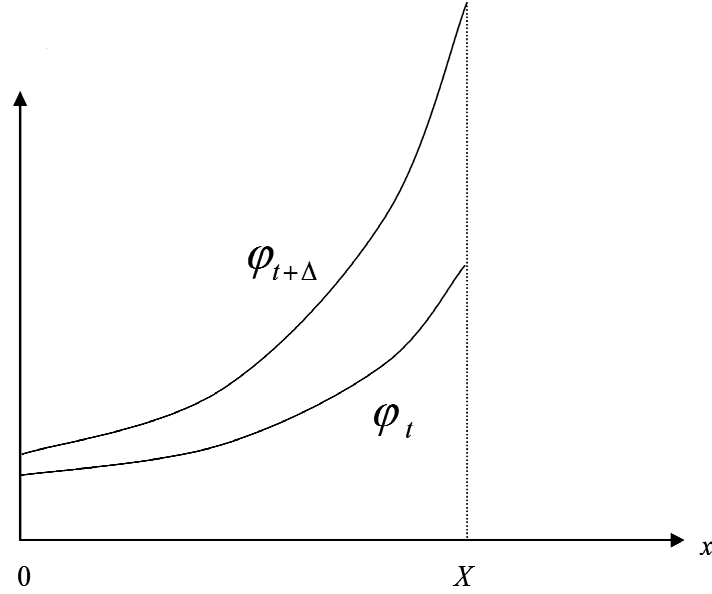
Online Appendices

1 Proofs and Technical Results

1.1 Characterization of the Production Process

The function $\varphi_t^*(x)$ is exponential in x and multiplicative in t , as depicted in Figure 1. As $t \rightarrow 0$ the function $\varphi_t^* : [0, X] \rightarrow \mathbb{R}$ converges to zero uniformly. As t grows, the function φ_t^* grows multiplicatively in t . Growth in t reflects a progressive increase in the number of active cases, that is, growing task juggling over time.

Figure 1: The path of the production process



Note. The figure depicts the distribution of active cases, by number of steps away from being done. On the growth path it is exponential.

Growth in task juggling also explains why the function $\varphi_t(x)$ is exponential in x . This is because, when the worker juggles an increasing number of projects over time, projects proceed at a progressively slower pace (that pace is η/A_t , and remember that A_t grows linearly with t). As projects grind along more and more slowly, the constant rate ν of newly inputted cases must squeeze in the progressively smaller “empty segment” available near X . This effect accounts for the exponential shape of $\varphi_t(x)$. Yet, remarkably, despite these complex dynamics the output rate is constant through time. This remarkable property of the output rate results from two opposite effects offsetting each other: on the one hand, cases move through at progressively slower rates, which tends to progressively reduce the output

rate. On the other hand, the mass of cases that are almost done increases with time (this is because $\varphi_t(0)$ grows with t), which tends to progressively increase the output rate. These two effects exactly offset each other along a constant growth path, and thus the output rate is time-invariant.

Theorem 1 goes a long way towards characterizing a constant growth path, but there is still some work to do. We need to characterize the relationship that links ν, η and ω along a growth path or, said differently, we need to understand what level of output is possible given certain input and effort rates. According to Theorem 1, the relationship between ν, η and ω along a growth path is given by equation (5). Define

$$h(y) = \frac{X}{\eta}y - \log(y).$$

Then equation (5) reads

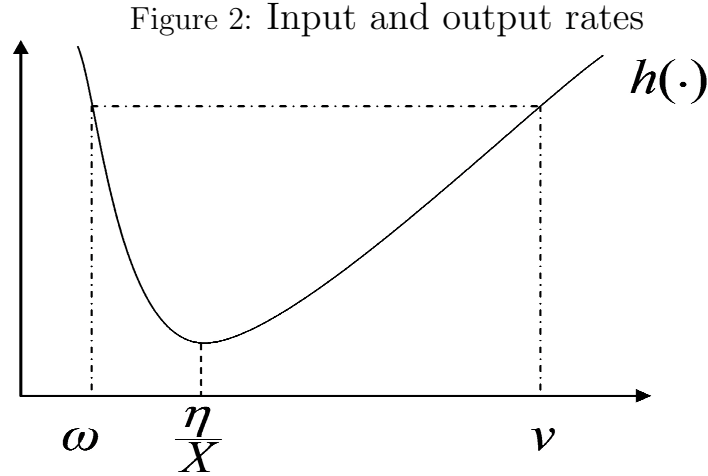
$$h(\nu) = h(\omega).$$

The next lemma characterizes the function $h(\cdot)$.

Lemma 1 *The function $h(y)$ is strictly convex on $(0, \infty)$, converges to infinity at $y = 0$ and $y = +\infty$, and it has its minimum at $y = \eta/X$.*

Proof. One can easily verify that $h(0) = +\infty = h(\infty)$, $h'(y) = \frac{X}{\eta} - \frac{1}{y}$, and finally $h''(y) = \frac{1}{y^2}$. ■

Figure 2 depicts $h(y)$. For a particular level of ν , the ω that solves equation (5) is represented



Note. The figure depicts the relationship between input and output rates on a growth path.

graphically as the point on the horizontal axis that achieves the same level of the function h . But not all solutions to equation (5) can be part of a growth path. Which solutions are consistent with a growth path is described in the next proposition.

Proposition 1 *ν, η and ω are related by (5) if and only if $\nu > \frac{\eta}{X}$. In that case, the ω generated by the pair $[\nu, \eta]$ is the unique solution that is smaller than $\frac{\eta}{X}$ to the equation $h(\nu) = h(\omega)$.*

Proof. The solution $\omega = \nu$ to equation (5) is not acceptable because then $A_t^* \equiv 0$ and (3) is not well-defined. Nor can we accept solutions where $\omega > \nu$, for then $\varphi_t^*(x)$ and A_t^* would be negative and thus the quadruple identified in Theorem 1 would not meet the definition of a growth path. So we need to find solutions with $\omega < \nu$. This implies $\nu > \frac{\eta}{X}$. The rest of the Proposition follows immediately from Theorem 1. ■

The threshold η/X can be interpreted as the minimum input rate compatible with the worker not being idle; we will discuss this interpretation at the end of this section. Proposition 1 shows how to construct the entire growth path associated with any pair (ν, η) . Given a constant input rate $\nu > \frac{\eta}{X}$, one can uniquely identify the corresponding output rate $\omega < \frac{\eta}{X}$ which solves $h(\nu) = h(\omega)$. Then the triple (ν, η, ω) is plugged into the expressions for $\varphi_t^*(x)$ and A_t^* to obtain a full characterization of the growth path.

Proposition 1 shows that our solution only makes sense if the input rate is sufficiently large. What happens otherwise? Then the worker can solve projects faster than she opens them, and in that case our model predicts $A_t \equiv 0$. In this case we do not have a model of task juggling, but rather one of “undercommitment.” We conclude this section by analyzing this case. In the analysis we allow for an “initial condition” $A_0 \geq 0$, a possibly positive mass of cases active at time zero. (This hypothesis deviates from our assumption that at $t = 0$ the mass of active cases is zero.) The next proposition shows that if $\nu < \frac{\eta}{X}$ then A_t shrinks over time, and if $\nu = \frac{\eta}{X}$ then A_t is constant.

Proposition 2 (*steady-state and shrink paths*) *If $\nu = \frac{\eta}{X}$ then there are a continuum of steady-state paths, indexed by the mass of projects active at time zero, A_0 . In each of these steady states $A_t \equiv A_0$, the output rate is equal to η/X , and the duration of projects is increasing in A_0 .*

If $\nu < \frac{\eta}{X}$ then whatever the value of A_0 , after a transition period it will be $A_t \equiv 0$ and, from then on, the duration of projects will be zero and the output rate will be equal to ν .

Proof. Let’s start with the case $\nu < \frac{\eta}{X}$. In this case the setup of the model described in Section 3 is no longer applicable, since that setup implicitly required that $A_t > 0$, which now

cannot be guaranteed. In fact, if we start at time 0 with $A_0 > 0$ and open projects at rate $\nu < \frac{\eta}{X}$, we expect $\omega_t > \nu$, and so we are on a temporary “shrink path” where over time A_t will shrink down to zero. After A_t hits zero, the worker completes projects instantaneously as soon as they are opened, and the system settles into a long-run path with $\omega_t = \nu < \frac{\eta}{X}$, and $A_t = C_t = D_t = 0$. In this long-run steady state, increasing ν increases ω contrary to Proposition 1.

In the case $\nu = \frac{\eta}{X}$, let us conjecture $\nu = \omega$ and so by (4) we have $A_t = A$. Fix any $A_0 > 0$. Note that this requires assuming an initial load of projects at time zero. Then (3) reads

$$\omega = \frac{\eta}{A_0} \varphi_t(0),$$

whence for all $t > 0$

$$\varphi_t(0) = \frac{A_0}{\eta} \omega. \tag{1}$$

Now, by definition we have that for all $x > 0$ we have $\varphi_t(0) = \varphi_\tau(x)$ for some $\tau < t$. This observation, together with (1), implies

$$\varphi_\tau(x) = \frac{A_0}{\eta} \omega \text{ for all } x, \tau.$$

Then (1) reads

$$A_0 = \int_0^X \varphi_t(x) dx = \frac{X}{\eta} A_0 \omega.$$

Note that this equality reduces to the identity $\omega = \eta/X$, which yields no new information. This means that any A_0 is compatible with the steady state path when $\nu = \eta/X$. Whatever is the initial condition of open projects A_0 , choosing $\nu = \eta/X$ will exactly perpetuate that mass of open projects.

The completion time of a newly opened project is the interval of time it takes the worker to process the A_0 projects that have precedence over it. We are looking for the time interval C_t it takes for a worker to complete A_0 projects. At a completion rate ω , C_t solves

$$\begin{aligned} A_0 &= \int_t^{t+C_t} \omega ds \\ &= \omega C_t = \frac{\eta}{X} C_t, \end{aligned}$$

whence the completion time of a newly activated project is $C_t = \frac{A_0}{\eta} X$, which is increasing

in A_0 . Given an arrival rate α , a project assigned at t finds

$$A_0 + \alpha t - \omega t$$

projects in front of it. The duration of a project assigned at t is the time it takes to complete these projects given an output rate ω . Thus the duration of a project assigned at t is also increasing in A_0 . ■

The threshold η/X can be interpreted as the minimum input rate compatible with the worker not being idle given that the worker exerts effort at rate η . To understand this interpretation, fix effort η and observe that if $\nu' < \eta/X$ then there exists a smaller effort rate η' such that $\eta'/X = \nu' \geq \omega'$ (the inequality is true because cannot be more cases being completed than there are coming in). This means that if the input rate ν' falls below η/X then the worker could achieve the same level of output ω' by exerting effort at the lower rate η' . This is equivalent to saying that the worker is idle at rate $\eta - \eta'$.

Proof of Proposition 1 a), b).

Proof. a) There are three types of solutions to the equation $h(\nu) = h(\omega)$. The first one is $\nu = \omega$. This solution is not compatible with the analysis we have carried out because then $A_t = 0$. Then there are two kinds of solutions, one where $\nu < \frac{\eta}{X} < \omega$, which is not acceptable for then $A_t < 0$. The remaining kind of solution is $\nu > \frac{\eta}{X} > \omega$. Under this restriction, the shape of $h(\cdot)$ guarantees the required property.

b) Fix ν , and consider two values $\eta > \eta'$ with associated ω and ω' . The output rates ω and ω' solve

$$\begin{aligned} h(\omega; \eta/X) &= h(\nu; \eta/X) \\ h(\omega'; \eta'/X) &= h(\nu; \eta'/X). \end{aligned}$$

Combining these equalities yields

$$h(\omega'; \eta'/X) - h(\omega; \eta/X) = h(\nu; \eta'/X) - h(\nu; \eta/X). \quad (2)$$

Now, an easy to verify property of $h(y; \eta/X)$ that, for any $y_1 < y_2$,

$$h(y_1; \eta'/X) - h(y_1; \eta/X) < h(y_2; \eta'/X) - h(y_2; \eta/X).$$

Setting $y_1 = \omega, y_2 = \nu$, and combining with (2) gives

$$\begin{aligned} h(\omega; \eta'/X) - h(\omega; \eta/X) &< h(\omega'; \eta'/X) - h(\omega; \eta/X) \\ h(\omega; \eta'/X) &< h(\omega'; \eta'/X) \end{aligned} \quad (3)$$

Now, remember that $\omega' < \eta'/X$. Then either $\omega > \eta'/X$, in which project $\omega > \omega'$ and there is nothing to prove, or else $\omega < \eta'/X$. In this project both ω and ω' lie on the decreasing portion of the function $h(\cdot; \eta'/X)$. Then equation (3) yields $\omega > \omega'$. ■

Next we prove a technical lemma that is necessary to prove Proposition 1 c).

Lemma 2 *Take any triple $(\nu, \omega, \frac{\eta}{X})$ where $\omega = \Omega(\nu; \eta/X)$. Then $|\nu - \frac{\eta}{X}| > |\omega - \frac{\eta}{X}|$. That is, along a growth path the **actual output** rate is closer to the **efficient output** rate than is the input rate.*

Proof. For any $\nu > \frac{\eta}{X} > \omega$ we can write

$$\begin{aligned} h(\nu) &= h\left(\frac{\eta}{X}\right) + \int_0^{\nu - \frac{\eta}{X}} h'\left(\frac{\eta}{X} + s\right) ds \\ h\left(\frac{\eta}{X}\right) &= h(\omega) + \int_0^{\frac{\eta}{X} - \omega} h'(\omega + r) dr. \end{aligned} \tag{4}$$

Make the change of variable $r = -\omega + \frac{\eta}{X} - s$ in the second equation, and one gets

$$\begin{aligned} h\left(\frac{\eta}{X}\right) &= h(\omega) - \int_{-\omega + \frac{\eta}{X}}^0 h'\left(\frac{\eta}{X} - s\right) ds \\ &= h(\omega) + \int_0^{-\omega + \frac{\eta}{X}} h'\left(\frac{\eta}{X} - s\right) ds. \end{aligned}$$

Substitute into equation (4) to get

$$h(\nu) = h(\omega) + \int_0^{-\omega + \frac{\eta}{X}} h'\left(\frac{\eta}{X} - s\right) ds + \int_0^{\nu - \frac{\eta}{X}} h'\left(\frac{\eta}{X} + s\right) ds.$$

Since the triple $(\nu, \omega, \frac{\eta}{X})$ solves (5), it follows that $h(\nu) = h(\omega)$ and so we may rewrite equation (4) once more as

$$\int_0^{\frac{\eta}{X} - \omega} -h'\left(\frac{\eta}{X} - s\right) ds = \int_0^{\nu - \frac{\eta}{X}} h'\left(\frac{\eta}{X} + s\right) ds \tag{5}$$

Now, from the proof of Lemma 1 we have $h'(y) = \frac{X}{\eta} - \frac{1}{y}$ and so

$$\begin{aligned} h'\left(\frac{\eta}{X} + s\right) &= \frac{X}{\eta} - \frac{1}{\frac{\eta}{X} + s} = \frac{X}{\eta} \left(1 - \frac{1}{1 + s\frac{X}{\eta}}\right) = \frac{X}{\eta} \left(\frac{s\frac{X}{\eta}}{1 + s\frac{X}{\eta}}\right) \\ h'\left(\frac{\eta}{X} - s\right) &= \frac{X}{\eta} - \frac{1}{\frac{\eta}{X} - s} = \frac{X}{\eta} \left(1 - \frac{1}{1 - s\frac{X}{\eta}}\right) = -\frac{X}{\eta} \left(\frac{s\frac{X}{\eta}}{1 - s\frac{X}{\eta}}\right) \end{aligned}$$

for any s such that $h'\left(\frac{\eta}{X} - s\right)$ is well defined, that is, $s < \frac{\eta}{X}$. If in addition $s > 0$ then

$$h'\left(\frac{\eta}{X} + s\right) = \frac{X}{\eta} \left(\frac{s\frac{X}{\eta}}{1 + s\frac{X}{\eta}}\right) < \frac{X}{\eta} \left(\frac{s\frac{X}{\eta}}{1 - s\frac{X}{\eta}}\right) = -h'\left(\frac{\eta}{X} - s\right). \quad (6)$$

Now let us turn to equation (5) and let us suppose, by contradiction, that $\nu - \frac{\eta}{X} < \frac{\eta}{X} - \omega$. We may then rewrite that equation as

$$\begin{aligned} &\int_0^{\frac{\eta}{X} - \omega} -h'\left(\frac{\eta}{X} - s\right) ds - \int_0^{\nu - \frac{\eta}{X}} h'\left(\frac{\eta}{X} + s\right) ds = 0 \\ &\int_0^{\nu - \frac{\eta}{X}} \left[-h'\left(\frac{\eta}{X} - s\right) - h'\left(\frac{\eta}{X} + s\right)\right] ds + \int_{\nu - \frac{\eta}{X}}^{\frac{\eta}{X} - \omega} -h'\left(\frac{\eta}{X} - s\right) ds = 0 \end{aligned}$$

The range of s in the above equation is at most $(0, \frac{\eta}{X} - \omega) \subset (0, \frac{\eta}{X})$, and therefore (6) applies. This guarantees that the first integral is strictly positive. The second integral is strictly positive as well. Hence the equation cannot be verified. We therefore contradict our assumption that $\nu - \frac{\eta}{X} < \frac{\eta}{X} - \omega$. ■

Proof of Proposition 1 c)

Proof. Equation (5) reads

$$(\nu - \Omega(\nu; \eta/X)) = \frac{\eta}{X} [\log(\nu) - \log(\Omega(\nu; \eta/X))]. \quad (7)$$

Fix ν and differentiate both sides of (7) with respect to η to get

$$-\frac{\partial \Omega(\nu; \eta/X)}{\partial \eta} = \frac{1}{X} [\log(\nu) - \log(\Omega(\nu; \eta/X))] - \frac{\eta}{X} \frac{1}{\Omega(\nu; \eta/X)} \frac{\partial (\Omega(\nu; \eta/X))}{\partial \eta}.$$

Rearranging we get

$$\frac{\partial \Omega(\nu; \eta/X)}{\partial \eta} \left[\frac{\eta}{X} \frac{1}{\Omega(\nu; \eta/X)} - 1 \right] = \frac{1}{X} [\log(\nu) - \log(\Omega(\nu; \eta/X))] \quad (8)$$

$$= \frac{1}{\eta} (\nu - \Omega(\nu; \eta/X)), \quad (9)$$

where the second equation substitutes from (7). Now, fix η and differentiate (8) with respect to ν . This yields

$$\begin{aligned} \frac{\partial^2 \Omega(\nu; \eta/X)}{\partial \eta \partial \nu} \left[\frac{\eta}{X} \frac{1}{\Omega(\nu; \eta/X)} - 1 \right] - \frac{\partial \Omega(\nu; \eta/X)}{\partial \eta} \frac{\eta}{X} \frac{1}{(\Omega(\nu; \eta/X))^2} \frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} \\ = \frac{1}{X} \left[\frac{1}{\nu} - \frac{1}{\Omega(\nu; \eta/X)} \frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} \right], \end{aligned}$$

which can be rewritten as

$$\frac{\partial^2 \Omega(\nu; \eta/X)}{\partial \eta \partial \nu} \left[\frac{\eta}{X} \frac{1}{\Omega(\nu; \eta/X)} - 1 \right] = \frac{1}{X} \left[\frac{1}{\nu} + \frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} \frac{1}{\Omega(\nu; \eta/X)} \left(\frac{\partial \Omega(\nu; \eta/X)}{\partial \eta} \eta \frac{1}{\Omega(\nu; \eta/X)} - 1 \right) \right]. \quad (10)$$

The term in brackets on the left-hand side is positive, so $\frac{\partial^2 \Omega(\nu; \eta/X)}{\partial \eta \partial \nu}$ has the same sign as the term in brackets on the right hand side of (10). We need to sign this term. To this end, substitute for $\frac{\partial \Omega(\nu; \eta/X)}{\partial \eta}$ from (9) so that the term in brackets on the right hand side of (10) reads

$$\begin{aligned} \frac{1}{\nu} + \frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} \frac{1}{\Omega(\nu; \eta/X)} \left[\frac{(\nu - \Omega(\nu; \eta/X))}{\left(\frac{\eta}{X} \frac{1}{\Omega(\nu; \eta/X)} - 1 \right) \Omega(\nu; \eta/X)} - 1 \right] \\ = \frac{1}{\nu} + \frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} \frac{1}{\Omega(\nu; \eta/X)} \left[\frac{\nu - \frac{\eta}{X}}{\frac{\eta}{X} - \Omega(\nu; \eta/X)} \right]. \end{aligned} \quad (11)$$

Now, to get an expression for $\frac{\partial \Omega(\nu; \eta/X)}{\partial \nu}$, fix η and differentiate both sides of (7) with respect

to ν to get

$$\begin{aligned}
1 - \frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} &= \frac{\eta}{X} \left[\frac{1}{\nu} - \frac{1}{\Omega(\nu; \eta/X)} \frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} \right] \\
\frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} \left[\frac{\eta}{X} \frac{1}{\Omega(\nu; \eta/X)} - 1 \right] &= \frac{\eta}{X\nu} - 1 \\
\frac{\partial \Omega(\nu; \eta/X)}{\partial \nu} &= \frac{\Omega(\nu; \eta/X)}{\nu} \frac{\frac{\eta}{X} - \nu}{\frac{\eta}{X} - \Omega(\nu; \eta/X)}.
\end{aligned}$$

Substituting into (11) yields

$$\begin{aligned}
&\frac{1}{\nu} - \frac{1}{\nu} \left(\frac{\nu - \frac{\eta}{X}}{\frac{\eta}{X} - \Omega(\nu; \eta/X)} \right)^2 \\
&= \frac{1}{\nu} \left[1 - \left(\frac{\nu - \frac{\eta}{X}}{\frac{\eta}{X} - \Omega(\nu; \eta/X)} \right)^2 \right].
\end{aligned} \tag{12}$$

By Lemma 2,

$$\frac{\nu - \frac{\eta}{X}}{\frac{\eta}{X} - \Omega(\nu; \eta/X)} > 1$$

and so equation (12) is negative. Thus the right hand side of (10) is negative, which implies $\frac{\partial \Omega(\nu; \eta/X)}{\partial \nu \partial \eta} < 0$. ■

Proof of Proposition 1 d),e)

Proof. d) Suppose the triple $(\nu, \omega, \frac{\eta}{X})$ solves (5). We need to show that for any scalar $r > 0$, the triple $(r\nu, r\omega, r\frac{\eta}{X})$ also solves (5), that is, that for any $r > 0$ we have

$$[\log(r\nu) - \log(r\omega)] = \frac{X}{r\eta} (r\nu - r\omega).$$

Write

$$\begin{aligned}
r \frac{\eta}{X} [\log(r\nu) - \log(r\omega)] &= r \frac{\eta}{X} [\log(\nu) - \log(\omega)] \\
&= r(\nu - \omega) = (r\nu - r\omega).
\end{aligned}$$

where the second equality follows because the triple $(\nu, \omega, \frac{\eta}{X})$ solves (5). The equality between the first and the last element in this chain of equalities shows that the triple $(r\nu, r\omega, r\frac{\eta}{X})$ solves (5).

e) Immediate from inspection of Figure 2. ■

Proof of Proposition 2.

Proof. (a) The completion time C_t of a project started at t is the time that it takes all the projects in front of it to clear. These projects are A_t , and given an output rate ω that duration is given by the solution to the following equation

$$\int_t^{t+C_t} \omega \, ds = A_t,$$

which equals

$$\omega C_t = (\nu - \omega) t.$$

Solving for C_t yields the desired expression. Let us now turn to duration. Given an arrival rate α , a project assigned at t finds

$$\alpha t - \omega t$$

projects in front of it. Given an output rate of ω , these projects will take

$$D_t = \frac{(\alpha - \omega)}{\omega} t$$

to complete. This is the duration of a project assigned at t .

(b) From part (a) we have

$$C_t = \left(\frac{\nu}{\Omega(\nu; \eta/X)} - 1 \right) t = \left(\frac{1}{\Omega(1; \eta/\nu X)} - 1 \right) t,$$

where the second equality follows from Proposition 1 d). From Proposition 1 (b) we have that Ω is increasing in its second argument, whence increasing ν decreases $\Omega(1; \eta/\nu X)$ and increases C_t .

As for duration, from part (a) we have

$$D_t = \left(\frac{\alpha}{\Omega(\nu; \eta/X)} - 1 \right) t.$$

From Proposition 1 (a) we have that Ω is decreasing in its first argument, whence increasing ν decreases $\Omega(\nu; \eta/X)$ and increases D_t . ■

1.2 Proofs for Section 4

The next lemma suggests that we should look for equilibria in which clients play just two simple strategies.

Lemma 3 *In any lobbying equilibrium in which the number of active projects grows, two strategies payoff-dominate all others: strategy $\mathbf{1}(\cdot)$ which denotes immediate and perpetual lobbying starting from time of assignment, and strategy $\mathbf{0}(\cdot)$ which denotes never lobbying.*

Proof. We prove that any strategy $S_\tau(\cdot)$ (typically displaying “intermittent” lobbying) is dominated either by strategy $\mathbf{0}(\cdot)$ or by strategy $\mathbf{1}(\cdot)$. Let us show this next. First, if $S_\tau(\cdot)$ is caught up, then it is dominated by the strategy $\mathbf{0}(\cdot)$ which achieves the same completion date at a lower lobbying cost. This is because after a strategy is caught up, it cannot go any faster than its assignment vintage. Suppose then that $S_\tau(\cdot)$ is not caught up.

Denote

$$\chi(t) = \int_{\tau}^t S_\tau(s) \, ds$$

where by construction $\chi(\cdot)$ is non-decreasing, $\chi(\tau) = 0$ and $\chi(t) \leq t - \tau$. The function $\chi(t)$ can be interpreted as a measure representing how much activity has occurred on the project between τ and t or, equivalently, the state of advancement of the project. When strategy S_τ is employed, the project’s advancement at time t is given by

$$\begin{aligned} x_S(t) &= X - \int_{\tau}^t \dot{x}_S(r) \, dr \\ &= X - \int_{\tau}^t \frac{\eta}{A_r} \, d\chi(r). \end{aligned}$$

Denote by T the time at which the project is done, that is, T is the smallest value that solves $x_S(T) = 0$. Create a new strategy $\tilde{S}(t)$ which equals 1 for $t \in [\tau, \tau + \chi(T)]$ and 0 for $t > \tau + \chi(T)$. Then we have

$$\begin{aligned} 0 &= x_S(T) \\ &= X - \int_{\tau}^T \frac{\eta}{A_r} \, d\chi(r) \\ &= X - \int_{\tau}^{\tau + \chi(T)} \frac{\eta}{A_{\chi^{-1}(y - \tau)}} \, dy \\ &\geq X - \int_{\tau}^{\tau + \chi(T)} \frac{\eta}{A_y} \, dy \\ &= X - \int_{\tau}^{\tau + \chi(T)} \frac{\eta}{A_y} \tilde{S}_\tau(y) \, dy = x_{\tilde{S}}(\tau + \chi(T)) \end{aligned}$$

where the third equality reflects a change of variable $y = \tau + \chi(r)$, and the inequality follows because $\chi(y) \leq y - \tau$, hence $\chi^{-1}(y - \tau) \geq y$ and $A_{\chi^{-1}(y - \tau)} \geq A_y$. The inequality shows that strategy S is just done at time T , whereas strategy \tilde{S} is more than done already by time

$\tau + \chi(T) \leq T$. This means that the duration under strategy \tilde{S} is smaller than that under strategy S . Denote by $\tilde{T} \leq \tau + \chi(T)$ the time strategy \tilde{S} is done. Let us now turn to lobbying expenditures. Strategy S 's lobbying expenditure is given by $\kappa\chi(T)$. Strategy \tilde{S} 's lobbying expenditure is given by $\kappa(\tilde{T} - \tau)$. Since $\tilde{T} \leq \tau + \chi(T)$, strategy \tilde{S} 's lobbying expenditure is smaller than strategy S 's.

Summing up, we have shown that duration and lobbying expenditure are smaller under strategy \tilde{S} than under strategy S . Thus strategy \tilde{S} dominates S . Notice that, since under \tilde{S} a project ends at $\tilde{T} \leq \tau + \chi(T)$, strategy \tilde{S} is payoff-equivalent to strategy $\mathbf{1}(\cdot)$. Thus strategy S is dominated by strategy $\mathbf{1}(\cdot)$. ■

The intuition behind Lemma 3 is the following. Lobbying “buys advancement” at the speed of η/A_t . If it is profitable to lobby at the assignment of the project, then it makes no sense to have interludes of no lobbying. During those interludes the project does not advance, but the mass of active projects A_t keeps growing, making lobbying (once it is restarted) less productive. Of course, even taking Lemma 3 into account, lobbying equilibria could potentially be very complex because of the possibility of non-constant growth equilibria in which the input rate is not constant through time.

Proof of Proposition 3

Proof. a) We show that there is a time-invariant z such that the value at the time of assignment of two players who follow the two different equilibrium strategies (lobby and not) are the same. The lobbyist's value at the time of assignment for a project assigned at τ , assuming the project is lobbied from assignment through to completion, is $(-\kappa - B)C_\tau$ where C_τ is the completion time of a project started at τ . Substituting for C_t from Proposition 2, the value is given by

$$VL_\tau(z) = (-\kappa - B) \left[\frac{\nu(z)}{\Omega(\nu(z); \eta/X)} - 1 \right] \tau.$$

The value of the non-lobbyist at the time of assignment for a project assigned at τ , assuming that she never lobbies, is computed as follows. First, the fraction of non-lobbyist projects inputted in each instant is given by $\frac{\nu}{\nu(z)}$, and consequently the output rate is made up of a fraction $\frac{\nu}{\nu(z)}$ of non-lobbyist projects. Thus, a project assigned at τ finds

$$z\alpha\tau - \frac{\nu}{\nu(z)}\Omega(\nu(z); \eta/X)\tau$$

non-completed projects in front of it. These projects are completed at rate $\frac{\nu}{\nu(z)}\Omega(\nu(z); \eta/X)$, so it takes

$$\left[\frac{z\alpha}{\frac{\nu}{\nu(z)}\Omega(\nu(z); \eta/X)} - 1 \right] \tau$$

before all non-lobbied projects assigned before τ are completed. Therefore the value of a non-lobbyist at the time of assignment, assuming that she never lobbies in the future, is

$$VN_\tau(z) = -B \left[\frac{z\alpha}{\frac{\nu}{\nu(z)}\Omega(\nu(z); \eta/X)} - 1 \right] \tau$$

In an equilibrium with lobbyists and non-lobbyists, z^* solves $VL_\tau(z^*) = VN_\tau(z^*)$, or

$$(-\kappa - B) \left[\frac{\nu(z^*)}{\Omega(\nu(z^*); \eta/X)} - 1 \right] = -B \left[\frac{\alpha}{\underline{\nu}} z^* \frac{\nu(z^*)}{\Omega(\nu(z^*); \eta/X)} - 1 \right] \quad (13)$$

It is important to note that condition is independent of τ . Thus, if a z^* exists that verifies equation (13), this z^* will be time-invariant, consistent with the definition of constant-growth lobbying equilibrium. We conclude the proof by showing that at least one z^* exists that verifies equation (13) and it lies between $\frac{\underline{\nu}}{\alpha}$ and $\frac{\underline{\nu}}{\alpha} + \frac{1}{\alpha}(\alpha - \frac{\eta}{X})$.

The lowest possible value of z^* is $\frac{\underline{\nu}}{\alpha}$. If z falls below this level, there are not enough non-lobbyists to fill $\underline{\nu}$, and then non-lobbied projects get started immediately. Formally, in this project the expression in brackets on the RHS of (13) is no greater than the brackets on the LHS, whence $VN_\tau(z) > VL_\tau(z)$. So $z \leq \frac{\underline{\nu}}{\alpha}$ is not consistent with equilibrium. The highest possible value of z^* is that for which $\nu(z^*) = \eta/X$. At this level the LHS of (13) is zero, and so $VN_\tau(z) < VL_\tau(z)$. Intuitively, if z^* were any higher, then $\nu(z^*) < \eta/X$ and then completion times would be zero, and then anyone who lobbied could do so at zero cost. Thus such z cannot be part of the equilibrium in which not everyone lobbies. To find an expression for this bound, write $\eta/X = \nu^* = \underline{\nu} + (1 - z)\alpha$, and solving for z yields $z = \frac{\underline{\nu}}{\alpha} + \frac{1}{\alpha}(\alpha - \frac{\eta}{X})$. We have shown that on the lower bound of the interval $z \in (\frac{\underline{\nu}}{\alpha}, \frac{\underline{\nu}}{\alpha} + \frac{1}{\alpha}(\alpha - \frac{\eta}{X}))$ we have $VN_\tau(z) > VL_\tau(z)$, and on the upper bound $VN_\tau(z) < VL_\tau(z)$. Since the two functions $VN_\tau(z)$ and $VL_\tau(z)$ are continuous in z over the interval, they must cross at least once. Any crossing is consistent with an equilibrium.

b) Suppose not, so that $\nu^* \leq \frac{\eta}{X}$. Then $\alpha > \nu^*$, and so a project assigned at τ finds a backlog of $(\alpha - \nu^*)\tau$ unopened projects in front of it. Since projects are opened at rate ν^* , the time it takes the last project in the backlog to be opened is

$$\frac{(\alpha - \nu^*)}{\nu^*} \tau.$$

This expression, which we will call the unopened duration, is positive and grows linearly with τ . This time can be eliminated by lobbying from assignment time all the way through completion, at a total cost that is proportional to completion time. Proposition 2 proves that when $\nu^* \leq \frac{\eta}{X}$ completion time is stationary, i.e., it is the same for projects opened at any τ . Therefore, the strict best response of all projects assigned after a certain $\hat{\tau}$ is to lobby all the way through completion time, in order to eliminate the unopened duration which exceeds lobbying costs. But then for every $t > \hat{\tau}$ not lobbying cannot be equally profitable as lobbying. Therefore we have shown that if $\nu^* \leq \frac{\eta}{X}$, a positive mass cannot be not lobbying after $\hat{\tau}$. Yet the construction requires that in any instant $\alpha - \nu^*$ projects are not lobbied, and this mass is positive because by assumption $\alpha > \frac{\eta}{X} \geq \nu^*$. Contradiction.

c) The equilibrium z^* solves (13), which can be rearranged as

$$\frac{\kappa + B}{B} \left[\frac{\nu(z)}{\Omega(\nu(z); \eta/X)} - 1 \right] = \left[\frac{\alpha}{\underline{\nu}} z \frac{\nu(z)}{\Omega(\nu(z); \eta/X)} - 1 \right]$$

and rewritten as

$$\left[\frac{\kappa + B}{B} - \frac{\alpha}{\underline{\nu}} z \right] \frac{\nu(z)}{\Omega(\nu(z); \eta/X)} = \left[\frac{\kappa + B}{B} - 1 \right]. \quad (14)$$

The LHS in (14) is the product of two positive and decreasing functions of z , and therefore it is decreasing in z . The RHS does not depend on z . Therefore equation (14) admits a unique solution z^* .

d) Rewrite slightly (14) as

$$H \left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}} \right) = \left[\frac{\kappa}{B} + 1 - \frac{\alpha}{\underline{\nu}} z \right] \frac{\nu(z)}{\Omega(\nu(z); \eta/X)} = \frac{\kappa}{B}. \quad (15)$$

The function $H \left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}} \right)$ is decreasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$, so increasing $\frac{\alpha}{\underline{\nu}}$ or $\frac{\eta}{X}$ results in a downward shift of the function. Since the function is decreasing in z , shifting the function downward results in a shift to the left of the intersection point between the function and the constant line $\frac{\kappa}{B}$. Thus z^* is decreasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$.

The function $H \left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}} \right)$ is increasing in $\frac{\kappa}{B}$, and increasing $\frac{\kappa}{B}$ by δ results in an upward shift of $\delta \frac{\nu(z)}{\Omega(\nu(z); \eta/X)} > 1$ in the function. So increasing $\frac{\kappa}{B}$ results in the function shifting upward by more than $\frac{\kappa}{B}$. So, start from a given $\frac{\kappa}{B}$ and focus on the resulting equilibrium z^* , which is the z at which the function H attains height $\frac{\kappa}{B}$. Then increase $\frac{\kappa}{B}$. At z^* , the function H moves up by more than $\frac{\kappa}{B}$. This means that z^* is to the left of the new equilibrium. Thus z^* is increasing in $\frac{\kappa}{B}$.

e) Follows directly from d) and the definition $\nu(z) = \underline{\nu} + (1 - z)\alpha$. ■

Proof of Proposition 4

Proof. Suppose by contradiction that, as κ increases to $\widehat{\kappa}$, we have $\widehat{z} < z^*$. Then by definition we have $\nu(\widehat{z}) > \nu(z^*)$. Since by Proposition 1 c) $\frac{\partial \Omega(\nu; \eta/X)}{\partial \nu \partial \eta} < 0$, it follows from problem (6) that $\widehat{\eta} < \eta^*$. Then $\Omega(\nu(\widehat{z}); \widehat{\eta}/X) < \Omega(\nu(z^*); \eta^*/X)$, and then the entire LHS of equation (15) becomes larger. Since the RHS stays unchanged, equation (15) can no longer be satisfied, and so we do not have an equilibrium. Therefore it must be that as κ increases to $\widehat{\kappa}$, the input rate decreases. It then follows from problem (6) that the worker's effort increases. ■

1.3 Proofs for Section 5

Define

$$F = f \frac{\nu - \omega}{\omega} t,$$

so that $F_t = F \cdot t$. Also define

$$\varphi_t^{**}(x) = \frac{(\nu - \omega) + F}{\eta} \omega t e^{\frac{(\nu - \omega) + F}{\eta} x},$$

and recall the previous definition

$$A_t^* = (\nu - \omega) t.$$

Proof of Proposition 5

Proof. (a) Condition (7) is verified because

$$\begin{aligned} & \frac{\partial \varphi_t^{**}(x)}{\partial t} - \frac{\partial \varphi_t^{**}(x)}{\partial x} \frac{\eta}{A_t + F_t} \\ &= \frac{(\nu - \omega) + F}{\eta} \omega e^{\frac{(\nu - \omega) + F}{\eta} x} - \frac{(\nu - \omega) + F}{\eta} \omega t e^{\frac{(\nu - \omega) + F}{\eta} x} \frac{(\nu - \omega) + F}{\eta} \frac{\eta}{(\nu - \omega + F) t} \\ &= \frac{(\nu - \omega) + F}{\eta} \omega e^{\frac{(\nu - \omega) + F}{\eta} x} - \frac{(\nu - \omega) + F}{\eta} \omega e^{\frac{(\nu - \omega) + F}{\eta} x} \\ &= 0. \end{aligned}$$

Condition (8) is verified because

$$\begin{aligned}
\omega &= \frac{\eta}{A_t + F_t} \varphi_t^{**}(0) \\
&= \frac{\eta}{(\nu - \omega + F)t} \frac{(\nu - \omega) + F}{\eta} \omega t \\
&= \omega.
\end{aligned}$$

Condition (4) can be verified immediately.

Condition (1) reads

$$A_t^* = \int_0^X \varphi_t^*(x) dx.$$

Substituting for $\varphi_t^{**}(x)$ and A_t^* yields

$$\begin{aligned}
(\nu - \omega)t &= \int_0^X \frac{(\nu - \omega) + F}{\eta} \omega t e^{\frac{(\nu - \omega) + F}{\eta} x} dx \\
&= \frac{(\nu - \omega) + F}{\eta} \omega t \int_0^X e^{\frac{(\nu - \omega) + F}{\eta} x} dx \\
&= \frac{(\nu - \omega) + F}{\eta} \omega t \left[\frac{\eta}{(\nu - \omega) + F} e^{\frac{(\nu - \omega) + F}{\eta} x} \right]_{x=0}^X \\
&= \omega t \left[e^{\frac{(\nu - \omega) + F}{\eta} x} \right]_{x=0}^X \\
&= \omega t \left[e^{\frac{(\nu - \omega) + F}{\eta} X} - 1 \right].
\end{aligned}$$

We can rewrite this equality as

$$\begin{aligned}
\frac{\nu}{\omega} - 1 &= \left[e^{\frac{(\nu - \omega) + F}{\eta} X} - 1 \right] \\
\frac{\nu}{\omega} &= e^{\frac{(\nu - \omega) + F}{\eta} X} \\
\log(\nu) - \log(\omega) &= (\nu - \omega + F) \frac{X}{\eta}
\end{aligned}$$

(b). Fix any ν and let ω^{**} be the output rate in a constant growth path with forgetful worker. Then ω^{**} solves equation (9), which can be written as

$$h(\omega) = h(\nu) + f \cdot \frac{\nu - \omega}{\omega} \frac{X}{\eta}. \quad (16)$$

Suppose $f > 0$, and by contradiction, that the output rate in a constant growth path with

non-forgetful worker, call it ω^* , is smaller than ω^{**} . Since obviously, $\omega^{**} < \nu$, we have $\omega^* \leq \omega^{**} < \nu$. By definition of ω^* we have $h(\omega^*) = h(\nu)$, and since the function h is convex, it follows that $h(\omega^{**}) \leq h(\nu)$. But then since $f > 0$, the right-hand side in (16) must exceed the left-hand side, and so the equation cannot be satisfied. We have reached a contradiction. ■

2 The data

We use data from one Italian court specialized in labor controversies for the industrial area of Milan. Our initial dataset contains all the 58,280 cases filed between January 1, 2000 and December 31, 2005. For 92% of these cases we have information on their entire history, while the remaining cases are observed up to December 3, 2007. These trials have been assigned to 31 judges who have been in service for at least one quarter during the period of observation. For the judges who were already in service on January 1, 2000, we also have information on the cases that were assigned to them in the previous year and we can therefore compute a measure of their backlog at the beginning of the period under study. For the judges who took service during the period of observation (or less than one year before January 1, 2000) we analyze their productivity starting from the fifth of their quarters of service, in order to give them time to settle in. All the cases assigned to them during the first year of service (including those that were transferred to them from previous judges who left for another office or retired) are nevertheless counted to compute their backlog at the beginning of the second year of service in which we start to analyze their productivity. Thus all the judges that we analyze have at least one year of tenure, and for each we know the backlog of not-yet-disposed cases at the beginning of the period of observation.¹³

We consider quarters as the relevant time unit and we focus on the subset of judges who received full workloads of new controversies within each quarter. We therefore eliminated the quarter observations concerning judges who did not receive a full workload because they retired, were transferred, were contemporaneously assigned to other duties or were in long term absence periods during which they were not receiving cases. At the end of this selection process, out of the original 31 judges we are left with the unbalanced panel of 21 judges. These judges, on average, receive 128 cases per quarter-judge, see Table 1 in this appendix.

In Italy, as in other countries, the law (Art. 25 of the Constitution) requires that judges receive a randomly assigned portfolio of new cases. This random assignment is designed to ensure the absence of any relationship between the identity of judges and the characteristics of the cases assigned to them. In the court that we consider the random assignment is implemented in the following way. Every morning the judges in service are ordered alphabetically starting from a randomly extracted letter of the alphabet. The cases filed during the day are then assigned in alphabetic sequence to all judges in service. In Coviello et al. (2012) we report the results of formal tests for the random allocation of cases to judges. These tests show that characteristics of the cases are independent of the identity of judges, who thus

¹³If a judge retires or is transferred to a different court (for whatever reasons) his/her cases are either all assigned to a new judge (thereby going in the initial backlog of the substitute) or they are distributed randomly to all the other judges in the court. We will later discuss the implications of these events for the econometric analysis.

receive, in the long run, qualitatively and quantitatively similar portfolios of controversies.

For each of the 21 comparable judges we compute three outcomes measures: duration of trials (the average number of days from filing until the date in which a sentence is deposited by the judge, or the case is settled, or censoring occurs in the 8 % of cases for which we do not see the end of the trial for cases assigned in the quarter); completion time (the average number of days elapsing between the date in which the first hearing is held and the date the case is completed for cases assigned in the quarter); throughput (the average number of cases closed in the quarter). We use the exact dates of scheduling of the hearings of a case, to compute two measures of task juggling: the number of active cases (the average number of cases which have received a first hearing but are not yet closed); the number of new opened cases (computed as the average number of cases which have received a first hearing in the quarter but are not yet closed). We leverage the random allocation of cases to judges to compute measures of ability (the average number of hearings to close a case assigned per unit of time) and effort (computed as the average number of scheduled hearings per unit of time in the quarter).

Descriptive statistics of the variables used in the analysis are displayed in Table 1.

Table 1: Descriptive statistics

	Mean	sd	p25	p50	p75	n
<i>Outcomes:</i>						
D : Total duration	290	77	229	285	352	381
C : Completion time	163	58	117	160	200	381
ω : Closed cases per quarter	119	35	98	122	145	381
<i>Task juggling:</i>						
A : Active cases at the end of quarter	210	74	154	206	266	381
ν : New opened cases per quarter	127	46	93	137	159	381
<i>Workload, effort, and ability:</i>						
α : New assigned cases per quarter	128	28	111	132	146	381
η : Hearings held in a quarter	396	125	306	425	490	381
X : Hearings to close cases	3.2	.57	2.9	3.3	3.6	381
$\frac{\eta}{X}$: Standardized effort per quarter	128	45	98	131	156	381

Note: All variables are defined as judges quarterly averages. *Total duration* is defined as the number of days elapsing between the filing date and the date in which the case is completed. *Completion time* is defined as the number of days elapsing between the date in which the first hearing is held and the date in which the case is completed. *Active cases at the end of the quarter* are defined as the number of cases which have received a first hearing but are not yet closed. *New opened cases per quarter* are defined as the number of cases that receive the first hearing in the quarter. *Standardized effort per quarter* is defined as the ratio between the number of hearings held by the judge in a quarter for any case, independently of the time of assignment, (*hearings held in a quarter*) and the number of hearings that were necessary to decide the cases assigned to the judge in the quarter (*hearings to close cases*).